Discount Rates for Seed Capital Investments

Abstract

So far, the estimation of discount rates required by entrepreneurs for seed capital investments has remained a mystery. Mongrut and Ramirez (2006) made a contribution to this area by deriving the lower bound discount rate for a non-diversified entrepreneur in an emerging market. However, they used a quadratic utility function, which does not have desirable assumptions. In this research one extends the previous work by deriving expressions of discount rates using a Hyperbolic Absolute Risk Aversion (HARA) utility function that includes the quadratic and the logarithmic forms as special cases. Furthermore, one also assumes the less risk-averse entrepreneur that invests almost all his capital in his project or firm and whose level of wealth approaches to zero. One finds that both expressions depend upon de entrepreneur's risk aversion parameter and a measure of the project total risk. By holding constant the risk free rate, one simulates the discount rates for the quadratic and the logarithmic forms. As expected, the entrepreneur's required returns (discount rates) are highly sensitive in both specifications and all of the values were lower than 50% and most of them were lower than 25%, but higher than the assumed risk free rate.

Keywords: Seed Capital, discount rates, entrepreneurship

JEL codes: L26 and M13

1. Introduction

In a pioneering work, McInish and Kudla (1981) argued that the so called two-fund separation theorem, originally developed by Tobin (1958), does not hold in the case of closely-held firms and small firms. Hence, the appropriate discount rate for valuating and investment project for these firms would be the owner's required rate of return instead of a market-derived discount rate.

The two-fund separation theorem states that the optimal portfolio of risky securities is exactly the same regardless of the investors' risk preferences[1]. This property is extremely important for asset pricing because it allows us to estimate functions to estimate the cost of equity capital under equilibrium conditions, such as in the case of the Capital Asset Pricing Model (CAPM). Recently, Breuer and Gürtler (2007) have shown that in order for the two-fund separation theorem to hold, two restricted conditions must be met, named, that the utility function must be always defined and the marginal utility must be positive (well defined problems). It happens to be the case that these well-defined problems restrict seriously the application of the two-fund separation theorem in practice.

This problem is even worse given the fact that not all preferences can be represented with one utility function. This is the main argument that Meyer (2007) advocates in order to specify marginal utility functions instead of the utility function itself because the former encompasses a bigger possible set of risk preferences.

Another avenue that the literature has pursued is to impose certain restrictions to the stock returns' distribution in order to attain the two-fund separation theorem. In this sense, Ross (1978) showed the conditions under which the theorem holds independently of the investor utility function, but Wei et.al. (1999) have shown that, although a multivariate normal distribution and in general elliptical distributions, can guarantee the two-fund separation property, the identification of the "true" return distribution is rather difficult, not to say impossible.

It is clear by now that using distribution properties of stock returns, is not a good avenue to guarantee the two-fund separation theorem. Furthermore, to follow the specification of utility functions that belong to the Hyperbolic Absolute Risk Aversion (HARA) family, only guarantee this separation property under restricted conditions.

The previous discussion is important if one is interested in obtaining the market value of an investment project. If this is not the case, the stock returns' distribution argument becomes irrelevant as the capital market is not going to be a benchmark anymore and the risk preference (utility function and marginal utility) argument becomes more relevant because it is the only way to specify the risk preferences and discount rates for seed capital projects and firms. Unfortunately, there is a cost of using subjective discount rates, named the estimated Net Present Value (NPV) turns out to be just one summary profitability indicator instead of being the normative investment rule (Zurita, 2005). In other words, two entrepreneurs could assess differently the same investment project. This is not possible under market-derived conditions such as the ones imposed by the Capital Asset Pricing Model (CAPM).

[1] Cas and Stiglitz (1970) managed to show the conditions under which the separation theorem holds under hyperbolic preferences explained later on in this paper.

Another problem related with the estimation of subjective discount rates is the possible bias in the estimation of the discount rate parameters. These different biases have been explained by Fuenzalida et al. (2007), so one must use a heuristic procedure in order to avoid as much as possible the potential biases.

Although, it is not possible to use a compelling investment rule and that there is the possibility of having biased estimations for the discount rate parameters, it is worth to derive one expression for the project's discount rate under certain assumptions whenever the discount rate parameters are estimated through a prospect and risk analysis processes focussed in devising strategies for increasing the success likelihood of the project rather than using just one indicator (i.e. NPV) to accept or reject the investment proposal.

In fact, Mongrut and Ramirez (2006) estimated this expression under quadratic risk preferences in an incomplete market. They concluded that, for the less risk-averse entrepreneur, the discount rate was not unique for an incomplete market, that it was bounded from below so only the maximum value for the investment project could be estimated and that the discount rate will depend upon three parameters: the risk-free rate, the project's reward-to-variability ratio (RTV) and the project total risk.

Mongrut et al. (2013) expanded the risk analysis process proposed by Fuenzalida et al. (2007) to estimate the project's total risk parameter. This expanded process includes a previous scenario analysis, so the project's NPV will be estimated for different scenarios and with the same RTV ratio, which implies the same investor's risk-aversion coefficient. In this way, one may estimate different project's total risk for the same project across the explicit time horizon within each scenario.

Although the discount rate expression provided by Mongrut and Ramirez (2006) is simple, it embeds unrealistic assumptions because they used a quadratic utility function that implies constant absolute risk aversion (CARA) and it is well-known that this utility function is not increasing everywhere[2]. The main goal of this research is to derive a discount rate expression whenever one uses HARA risk preferences. It turns out to be that the discount rate expressions depend upon a measure of the project's total risk and the risk aversion of the entrepreneur.

The paper is organized as follows, the next section defines the concept of a discount rate according to the features of seed capital investments, then in the third section one explains briefly all the new set of utility functions available and justifies the use of the HARA family. In the fourth section one derives an expression for the discount rate using the HARA family and assuming complete markets. In the fifth section, with the aid of simulation, one explores the properties of the discount rate expressions. In the last section one concludes the work.

[2] The three traditional assumptions when using utility functions are: the utility function is an increasing function everywhere of consumption or wealth (meaning that more is better), the utility function is concave and twice differentiable, where the first derivative is positive (marginal utility) and the second is negative (Gerber and Pafumi, 1999).

2. Definition of a discount rate in the context of seed capital investments

Very often the discount rate is being defined as the rate at which one discount future cash flows to the present. This is really not a definition, but a description of its application. In this section one not only defines the discount rate, but also, one discusses the different features that a utility function must encompass in order to be applied in the context of seed capital investments.

2.1 Time preferences, time value of money and risk

There are two parallel strands of literature discussing issues about the estimation of the discount rate: the social choice literature and the private choice literature. Although, in both cases, the discount rate is being used to bring back to the present future net benefits, what it represents is quite different.

In terms of the social choice literature what matters is the time preference problem that is the trade-off between present and future consumption. Since this trade-off is being made on an individual basis, then one needs to aggregate these individual benefits into a measure of social benefits (Cameron and Gerdes, 2003). Hence, much of the literature has been devoted to estimate individual discount rates using exponential, hyperbolic or generalized hyperbolic models, and surveys and simulations to validate them. The main conclusions so far is that individuals' opinions about social discount rates vary substantially across samples, across the choice contexts and across different techniques used to elicit them. The main problem is that most individuals do not know the meaning of a discount rates and if they know it, nothing guarantees that they able to exteriorize the magnitude of their individual discount rate.

From the social choice literature, it seems that there is a urgent need to gather only experts opinions and to translate the language of a discount rate into its constituents that, hopefully, are far more familiar to experts rather than to common individuals.

In the case of the private choice literature, one is faced with only one choice: to invest or not to invest in a certain investment project. However, there is also the problem of social aggregation that needs to be solved. The discount rate in this context is understood as an opportunity cost that considers not only the time value of money, but that it also includes the risk that the investor is facing. In other words, it is an opportunity cost suitable or comparable to the project's risk level.

Here, there are two common interpretations that lead to market-derived discount rates or to subjective discount rates. In the former case, it is being assumed that each investor holds a well-diversified portfolio of investments; hence he only needs to worry about the project's market risk, which is the project's contribution to the risk of this well-diversified portfolio. In the latter case, one assumes that investors do not hold a portfolio of investment projects and in this way they are completely exposed to the project's total risk. Naturally, these are two extremes assumptions from a range of possible degrees of diversification, but they are quite convenient because they can make the models more tractable.

The aggregation problem in the case of market-derived discount rates is solved by using the two-fund separation theorem. Specifically, one postulates the existence of a representative well-diversified investor, so each one of the assets is priced according to its market risk. Finally, by using the market-derived discount rate one is estimating the value of the asset from the market's point of view (the representative investor). In the case of individually-derived discount rates the aggregation problem is of a different nature because here one only needs to obtain a consensus forecast of the experts' opinion. However, nothing guarantees that this consensus forecast is valid for the market as a whole. In fact, it is only helpful to avoid biases in the estimation of the project critical variables.

If one considers risk as the degree of uncertainty associated with gaining a competitive advantage, hence, the market risk is the relevant one for well-diversified investors and the total risk is the relevant one for non-diversified investors that are usually the owners of seed capital projects and firms (Erikson, 2005). Hence, it is of paramount importance to estimate properly both risks in these two extreme cases.

2.2 The utility function for seed capital investments

In a seminal work, McMahon and Stanger (1995) discussed different factors that affect small firms' objective function. They argued that the owner-manager's utility function depends upon pecuniary returns from the business (P) such a return on investment (ROI); non-pecuniary satisfactions from the business that are in the financial domain (Nf) such a good financial health, having operative flexibility, and so on; non-pecuniary satisfactions from the business that are outside the financial domain (Nn) such as preferred lifestyle, having self-esteem, and so on; and total risk, which is the sum of systematic or market risk and unsystematic or enterprise-specific risk (ϑ). Hence, the utility function for a small firm's owner-manager will be as follows:

$$U(w) = U(P, N_f, N_n, \vartheta) \tag{1}$$

In this specification, clearly the owner-manager's wealth is affected by all the factors already described. Furthermore, these authors went further explaining why the firms' return, firm's risk, firm's liquidity, the owner-manager's degree of diversification, the transferability of financial and human capital, the financial and investment flexibility, the desire of control, and the owner-manager's accountability would impact this utility function through their different variables.

Although this utility function seems plausible as a generalization for small firms, two comments are in place: there are many types of small firms and hence some of them are going to be less affected by some factors, and most of these factors influence also the utility function of a corporate firm. The point here is not the difference between sizes among firms, but the different relative impacts of each of these factors in the firm's (entrepreneur) utility function. For instance, the possibility to transfer financial and human capital from one generation to the other is important for certain types of small firms such as family firms. In this case, the founder derives part of his satisfaction by looking at how his heirs takeover the business properly. However, for the owner-manager of a star-up firm it is more important to have financial and investment flexibility because he needs funds, which are usually not provided by venture capitalists in the early stage of the business. In other words, he must be able to show to potential creditors or shareholders the start-up firm is able to cope with different scenarios, so that it has a reasonable likelihood of succeeding.

Other factors, such the firm's return, liquidity, desire of control and accountability are also important for corporate firms. In fact, there is no way in which a company can survive in the long-term if it does not have liquidity and if it does not care for the firm's stakeholders. Of course, here one must recognize that liquidity could be a matter of life and death for a small firm, but this means that one must recognise this factor somehow in the investment analysis and that it must emerge as a "critical one" in a particular case, if not, it is of no more importance than for a corporate firm.

The owner-manager's degree of diversification and the firm's risk are crucial because they change the valuation tools for small firms and in particular for seed capital projects and firms. A seed capital project or firm is usually undertaken or run by a single non-diversified entrepreneur that faces the project's total risk. A market-derived valuation framework does not apply here because there is no representative investor that matters, only matters the entrepreneurs and the experts' opinions concerning a particular investment proposal or firm. Besides, since the entrepreneur usually puts all his money in the project or firm, so he will bear the project's total risk. Furthermore, in this case the traditional value-additivity principle breaks up, because the entrepreneur could derive some positive synergies and diversification by investing in different projects[3].

Another issue worth to discuss is whether to include or not in the utility function the non-pecuniary satisfactions that do not lie within the financial domain (Nn). Although it is important to have a preferred lifestyle, being secure in self-employment, to offer secure employment to family and friends, having self-esteem, and so; one must recognize that most of these satisfactions do not apply for all small firms and some of them are really intrinsic and beyond the firm's fate, so there is the question to include them or not in the entrepreneur's utility function.

For instance, having a good self-esteem is a personal attribute that a person must cultivate regardless on whether he becomes an entrepreneur or not. Of course, if he is interested to become an entrepreneur, as a preferred lifestyle, he will derive an important satisfaction by becoming a successful one, but also by failing and learning. One could argue that only entrepreneurs who have failed are more able to reach and flavour the sweet side of becoming successful.

In other words, if having self-esteem and becoming an entrepreneur by conviction are two attributes beyond the firm's fate, does it worth to value the marginal extra satisfaction that the entrepreneur may obtain through his entrepreneurial adventure? Since he will derive a positive satisfaction anyway by running the business, one may think about these non-pecuniary satisfactions as lying in a superior indifference curve, but all with the same slope. In other words, it is like having a constant that moves the utility function upward, but these non-pecuniary satisfactions do not restrict the shape and the optimum solution. Since the optimum would not be affected, hence one may decide not to include this constant in the entrepreneur's utility function[4].

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3 The value-additivity principle states that the value of a firm, for a well-diversified investor, is equal to the value of the firm without the project plus the NPV of the project. This is possible because only the firm's contribution to the well-diversified investor's portfolio matters, so any potential synergy between the project and the firm does not have a market value because the investor could diversify better and quicker than the firm. In the case of a non-diversified entrepreneur, this principle does not hold.

4 This decision amounts to assume an entrepreneur who is self-confident, who decided to be an entrepreneur by conviction and who runs his business in a strictly professional way.

Concerning the non-pecuniary satisfactions related to the financial domain (Nf), they are better approached by analyzing the investment project trough different scenarios and devising strategies that improve its financial health and likelihood of success. Of course one must also include them in the entrepreneur's utility function and derive a discount rate accordingly, but one can also include them by analyzing the firm's critical variables and cash flows through different scenarios[5].

In brief, one believes that the utility function for seed capital investments must be a subset of the utility function described by McMahon and Stanger (1995):

$$U(w) = U(P, \vartheta) \tag{2}$$

Note that "P" is related to pecuniary returns from the business, especially the return on investment (ROI) that helps to provide a good salary for owner-manager. Given that most entrepreneurs or owner-managers of seed capital projects or firms, respectively, are not diversified, the second parameter (ϑ) is related to the project or firm's total risk. The non-pecuniary satisfactions related to the financial domain (Nf) are addressed directly in the scenario and risk analysis processes and the pecuniary satisfactions outside of the financial domain (Nn) are not considered in the entrepreneur or owner-manager's utility function.

3. The choice of the utility function for discount rates

The field of risk preferences (utility functions) was foster by the work of Arrow (1965, 1971) and Pratt (1964). In particular, they both proposed the following specification of utility functions[6]:

$$U(C_t) = -e^{-\alpha C_t} \tag{3}$$

This function is well-known as the negative exponential and it has a constant absolute coefficient of risk aversion ($A(c_t) = \alpha$). They also proposed the following specification[7]:

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{4}$$

This utility function is known as the power utility function where the parameter (γ) represents the level of relative risk aversion. Unfortunately, both specifications only let us vary the magnitude of the risk aversion (absolute or relative), but they do not allow us to change the slope of these risk aversion measures (Meyer, 2007).

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5 For example, James (1999) has included the transferability issue into the owner-manager's utility function of a family firm. Interestingly, he founds that this issue could extend the owner-manager's investment horizons.

6 One can also express the utility function in terms of wealth (W) without loss of generality.

7 The coefficient of relative risk aversion (CRRA) is equal to $R(c_t) = c_t A(c_t)$ and it implies that risk aversion also depends upon the individual's level of consumption, which in turn depends on his wealth.

The slopes represent the behaviour of the individual's risk aversion because it may depict decreasing (or increasing) absolute risk aversion (DARA or IARA) .In one hand, a DARA behaviour implies that richer people are less absolutely risk averse than poorer people, so they require a smaller payment than a richer people. In the other hand IARA behaviour from the side of richer people implies that they require a larger payment than poorer people in order to enter in a lottery game. Naturally, the DARA case is more realistic than the IARA case.

In 1971, Merton proposed a family of utility functions named Hyperbolic Absolute Risk Aversion (HARA) that has two big advantages: it includes the previous specifications as particular cases and it includes a wider range of risk preference specifications. Furthermore a particular choice of the function parameters can change the slopes of the risk aversion coefficients (CARA and CRRA):

$$U(C_t) = \frac{\gamma}{1-\gamma}\left[\mu + \frac{\alpha C_t}{\gamma}\right]^{1-\gamma} \tag{5}$$

Where:

$\alpha > 0$; $\mu + \frac{\alpha C_t}{\gamma} > 0$; and $C_t > 0$

For instance, if $\gamma > 0$ then absolute risk aversion is decreasing, while if $\gamma < 0$ then absolute risk aversion is increasing. If $\alpha = \infty$ then the coefficient of relative risk aversion will be equal to γ.

If $\gamma \to \infty$ then $\mu = 1$ and the so called generalized power utility function appears:

$$U(C_t) = \frac{\gamma}{1-\gamma}\left[1 + \frac{\alpha C_t}{\gamma}\right]^{1-\gamma} \tag{6}$$

From this specification, one could obtain the negative exponential function, the logarithmic utility function and the quadratic utility function:

If $\gamma \to 0$ and $\alpha > 0$, then the generalized power utility function (expression 6) converges to the negative exponential function given by expression 3 and it has an absolute risk aversion coefficient equal to α.

If $\gamma \to 1$ then an affine transformation of expression 6 converges to the logarithmic utility function:

$$U(C_t) = Ln\left(C_t + \frac{1}{\alpha}\right)$$ Where: $C_t + \frac{1}{\alpha} > 0$

This specification has a decreasing absolute risk aversion coefficient equal to $A(C_t) = \frac{1}{C_t + \frac{1}{\alpha}}$. If $\alpha = \infty$ then the relative risk aversion coefficient is equal to 1.

If $\gamma \to -1$ then an affine transformation of expression 6 converges to the quadratic utility function[8]:

$$U(C_t) = -\frac{1}{2}\left(\frac{1}{\alpha} - C_t\right)^2 \text{ Where: } \frac{1}{\alpha} - C_t > 0$$

This specification has an increasing absolute risk aversion coefficient equal to $\frac{1}{\frac{1}{\alpha} - C_t}$.

Recently, some new utility functions have been put forward in the literature. For instance Saha (1993) introduced the expo-power (EP) utility function. The functional form is as follows:

$$U(C_t) = \vartheta - e^{-\beta C_t^{\alpha}}$$

With the following restrictions: $\vartheta > 1$ and $\alpha\beta > 0$

Given these restrictions, the absolute risk aversion coefficient is equal to the following expression: $A(C_t) = \frac{1 - \alpha + \alpha\beta C_t^{\alpha}}{C_t}$. As Meyer (2007) pointed out, this is a two parameter model because the constant "ϑ" does not enter the risk aversion coefficient.

The main advantage of the EP functional form over the HARA is that the EP form reduces to the CARA form with finite parameter values. Remember that in the HARA form (expression 6) one needs to assume that $\gamma \to 0$ in order to have the CARA form. Xie (2000) tried to improve also a functional form over the HARA function, so he proposed the Power Risk Aversion (PRA) functional form. The advantage of this new functional form over the HARA is that it remains well defined over a longer parameter values' span (Meyer, 2007):

$$U(C_t) = \frac{1}{\gamma}\left\{1 - \exp\left[-\gamma\left(\frac{c_t^{1-\sigma} - 1}{1-\sigma}\right)\right]\right\} \text{ With the following restrictions: } \sigma \geq 0 \text{ and } \gamma \geq 0$$

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8 The affine transformation of expression 6 is the following: $U(C_t) = \frac{\gamma}{1-\gamma}\left[\left(\frac{1}{\alpha} + \frac{C_t}{\gamma}\right)^{1-\gamma} - 1\right]$

In the previous expression the exponential operator (exp) refers to the transcendental number "e". The absolute risk aversion coefficient is given by the following expression:

$$A(C_t) = \frac{\sigma}{C_t} + \gamma C_t^{-\sigma}$$

As Meyer (2007) points out the EP and PRA forms are the same. In fact, the improvement over the HARA functional form is of a similar nature, to have either a well-defined constant absolute risk aversion coefficient or a well-defined functional form over a longer parameters values' span. Finally, Connife (2006) has provided the Flexible Three Parameter (FTP) functional form that encompasses the EP and PRA forms. The functional form is given by the following expression:

$$U(C_t) = \frac{1}{\gamma}\left\{1 - \left[1 - k\gamma\left(\frac{C_t^{1-\sigma} - 1}{1-\sigma}\right)\right]^{\frac{1}{k}}\right\}$$ When k=0 it reduces to the PRA (Meyer, 2007)

The absolute risk aversion coefficient is given by the following expression:

$$A(C_t) = \frac{\sigma}{C_t} + \frac{(1-k)\gamma C_t^{-\sigma}}{1 - k\gamma\left(\frac{C_t^{1-\sigma} - 1}{1-\sigma}\right)}$$

Despite the fact that using an HARA utility function just captures a more restrictive range of investors' preferences than the EP, PRA and FTP forms, it is by far the most used closed-form specification in financial economics. For instance, the HARA utility functions have been applied for estimating discount rates in the case of bankruptcy (Barthelemy et. al., 2006), for portfolio performance evaluation (Breuer and Gürtler, 2006), for portfolio allocation with hedge funds (Popova, 2006), and for expanding market-derived conditions for pricing under incomplete markets (Menoncin (1998), Luenberger (2002), and Guu and Wang (2008)).

If one aims to capture the broadest types of risk preferences, it would be better to follow the approach of Meyer (2007). He's approach is based upon defining a closed-form function for the marginal utility function instead of the utility function itself. This is because, the marginal utility also represents a risk preference and not all marginal utility has a closed-form utility function. Even so, not all risk preferences can be captured into closed-forms marginal utilities.

In this work, one believes that is possible to use a quite common and broad closed-form utility function to estimate the lower bound of a discount rate and then, trough a process of scenario and risk analysis, one may estimate the resulting expression's parameters. In fact, Mongrut and Ramirez (2006) have shown that, under incomplete markets, there are many possible discount rates so the possibilities to specify different values for the parameters are enormous, what matters is how one may characterize the project's critical variables given the underlying rationale to move from scenario to another one.

4. Discount rates for seed capital investments under incomplete markets

In this section, one uses an affine transformation of expression 6 to solve the problem stated by Mongrut and Ramirez (2006): how much to consume today and tomorrow given that each individual has an initial wealth and each one could invest in one investment project or make a deposit in a savings account. However, in order to put the problem more realistic, one analyses the case when the entrepreneur's initial wealth level approaches to zero, given that he is the less-risk averse and he invest all his money in the project or firm.

It is important to state beforehand that the solution this problem can be considered a generalized solution only the in the sense that it accounts for a broader set of risk preferences rather than the quadratic utility function used by Mongrut and Ramirez (2006). However, it should be clear by now that using expression 6 is only a subset of all possible risk preference' specifications. All the remaining assumptions made by Mongrut and Ramirez (2006) are also made in this derivation namely, it is an individual optimization meaning that each individual must define the parameters of the resulting discount rate expression according to his risk aversion and the specifics of the project. However, one can also define the lower bound expression assuming the less risk-averse entrepreneur, which implies an absolute risk aversion coefficient equal to 1. One starts by optimizing the following two-period objective function (one also assumes that the utility function is time separable):

$$U(C_t, C_{t+1}) = \frac{\gamma}{1-\gamma}\left[\left(1+\frac{\alpha C_t}{\gamma}\right)^{1-\gamma} - 1\right] + \delta E_t\left\{\frac{\gamma}{1-\gamma}\left[\left(1+\frac{\alpha C_{t+1}}{\gamma}\right)^{1-\gamma} - 1\right]\right\}$$

Subject to the following constraint: $E(R_{t+1}) = \frac{C_{t+1}}{W_t - C_t}$

Where:

$E(R_{t+1})$: Represents the project expected return in period "t+1"

W_t: The entrepreneur initial wealth level.

C_t: The entrepreneur's consumption from today

C_{t+1}: The entrepreneur's consumption from tomorrow

The stochastic discount factor (SDF) of the previous problem is equal to:

$$m_{t+1} = \delta\left(1+\frac{\alpha C_t}{\gamma}\right)^{\gamma}\left(1+\frac{\alpha C_{t+1}}{\gamma}\right)^{-\gamma} \qquad (7)$$

In order to estimate the parameters of the SDF for this individual, one may state the following system of equations (Mongrut and Ramirez, 2006):

$$1 = E_t(m_{t+1}R_{t+1})$$

$$\frac{1}{R_f} = E_t(m_{t+1})$$

Using the SDF from equation 7, this system of equation translates into the following system:

$$1=\delta\left(1+\frac{\alpha C_t}{\gamma}\right)^{\gamma} E_t\left[\left(1+\frac{\alpha R_{t+1}(W_t-C_t)}{\gamma}\right)^{-\gamma} R_{t+1}\right] \tag{8}$$

$$\frac{1}{R_f}=\delta\left(1+\frac{\alpha C_t}{\gamma}\right)^{\gamma} E_t\left[\left(1+\frac{\alpha R_{t+1}(W_t-C_t)}{\gamma}\right)^{-\gamma}\right] \tag{9}$$

From equations 8 and 9, it follows the following condition:

$$0=E_t\left[(-R_f+R_{t+1})\left(1+\frac{\alpha R_{t+1}(W_t-C_t)}{\gamma}\right)^{-\gamma}\right] \tag{10}$$

$$0\approx E_t\left[\sum_{k=0}^{\infty} -Binomial(-\gamma,k)(R_f-R_{t+1})R_{t+1}^{k}\left(\frac{\alpha(W_t-C_t)}{\gamma}\right)^{k}\right] \tag{11}$$

Simplifying even more expression 11 it yields:

$$E_t(R_{t+1})-R_f\approx E_t\left[\sum_{k=1}^{n} Binomial(-\gamma,k)(R_f-R_{t+1})R_{t+1}^{k}\left(\frac{\alpha R_{t+1}(W_t-C_t)}{\gamma}\right)^{k}\right] \tag{12}$$

Expression 12 is the most important one because from it one derives the expressions of the discount rates for two risk preferences: quadratic and logarithmic. The discount rate corresponding to the case of a negative exponential utility function is not possible to obtain because $\alpha \neq 0$ and 1. One must remember that the constant absolute risk aversion coefficient of this utility function is equal to α.

The case of a quadratic utility function appears when the following conditions are met: $\gamma \rightarrow -1$, k = -1 and $\alpha > 0$. By replacing these conditions in equation 12 yields:

$$E_t(R_{t+1})-R_f\approx E_t\left[\sum_{k=-1}^{-1} Binomial(1,-1)(R_f-R_{t+1})R_{t+1}^{-1}\left(\frac{\alpha R_{t+1}(W_t-C_t)}{-1}\right)^{-1}\right] \tag{13}$$

Simplifying the previous expression yields:

$$E_t(R_{t+1})-R_f\approx E_t\left[\frac{R_{t+1}-R_f}{R_{t+1}(W_t-C_t)\alpha}\right] \tag{14}$$

Now, one must consider the following two conditions. The first one is related to the less risk-averse investor and the second to the fact that early-stage entrepreneurs start with a small initial level of wealth.

$$A = \frac{1}{\frac{1}{\alpha} - C_t} = 1 \Rightarrow C_t = \frac{1-\alpha}{\alpha}$$

$$W_t \to 0$$

Given these two conditions, equation 14 provides the lower bound for the discount rate for entrepreneurs with quadratic preferences:

$$E_t(R_{t+1}) \geq R_f + \left(\frac{1}{\alpha - 1}\right) RTV(CV_t) \tag{15}$$

Where:

$RTV = \dfrac{E_t(R_{t+1}) - R_f}{\sigma(R_{t+1})}$ is the reward-to-variability ratio and

$CV_{t+1} = \dfrac{\sigma(R_{t+1})}{E_t(R_{t+1})}$ is the coefficient of variation

However, for the less risk-averse entrepreneur, one may assume that $RTV \approx 1$, hence equation 15 boils down to:

$$E_t(R_{t+1}) \geq R_f + \left(\frac{1}{\alpha - 1}\right)(CV_{t+1}) \tag{16}$$

Note that in this case, the discount rate has three parameters: the risk-free rate (Rf), the coefficient that measure the magnitude of the entrepreneur risk-aversion ($1/_{\alpha - 1}$) and the coefficient of variation that comes from the seed capital project. Besides, the absolute risk aversion coefficient is increasing because $\gamma < 0$ and $\alpha > 1$.

Alternatively, one may study the case of the logarithmic risk preference. This situation appears when the following conditions are met: $\gamma \to 1$, k = 1 and $\alpha > 0$. By replacing these conditions in equation 12 yields:

$$E_t(R_{t+1}) - R_f \approx E_t\left[\sum_{k=1}^{1} Binomial(-1,1)(R_f - R_{t+1})R_{t+1}^{1}\left(\frac{\alpha R_{t+1}(W_t - C_t)}{1}\right)^{1}\right] \tag{17}$$

Simplifying this expression taking into account the following property yields expression 18:

$$E_t\left[R_{t+1}^2\right] = \sigma^2(R_{t+1}) + E_t[R_{t+1}]^2$$

$$E_t[R_{t+1}] - R_f \approx \frac{\alpha(W_t - C_t)CV_t\sigma(R_{t+1})}{CV_t - \alpha(W_t - C_t)\sigma(R_{t+1})}\sigma(R_{t+1}) \quad (18)$$

Again in expression 18 one needs to consider the following two additional restrictions (the entrepreneur is the less risk-averse one and his level of wealth approaches to zero):

$$A = \frac{1}{\frac{1}{\alpha} + C_t} = 1 \Rightarrow C_t = \frac{\alpha - 1}{\alpha}$$

$$W_t \to 0$$

Replacing these two restrictions in equation 18, yields the lower bound discount rate for an entrepreneur with a logarithmic risk preference[9]:

$$E_t[R_{t+1}] \geq R_f + \left[\frac{(1-\alpha)\sigma^2(R_{t+1})}{CV_{t+1} - (1-\alpha)\sigma(R_{t+1})}\right]CV_{t+1} \quad (19)$$

Note that in this case the discount rate has also three parameters, but the risk-aversion parameter confounds the "α" risk-aversion coefficient with the project's total risk. This specification implies a decreasing risk-aversion coefficient and $0 < \alpha < 1$.

5. Properties of the discount rate expressions

In this section one identifies the critical parameters in both expressions for the entrepreneur's lower bound discount rate (expressions 16 and 19) and then one performs a simulation analysis.

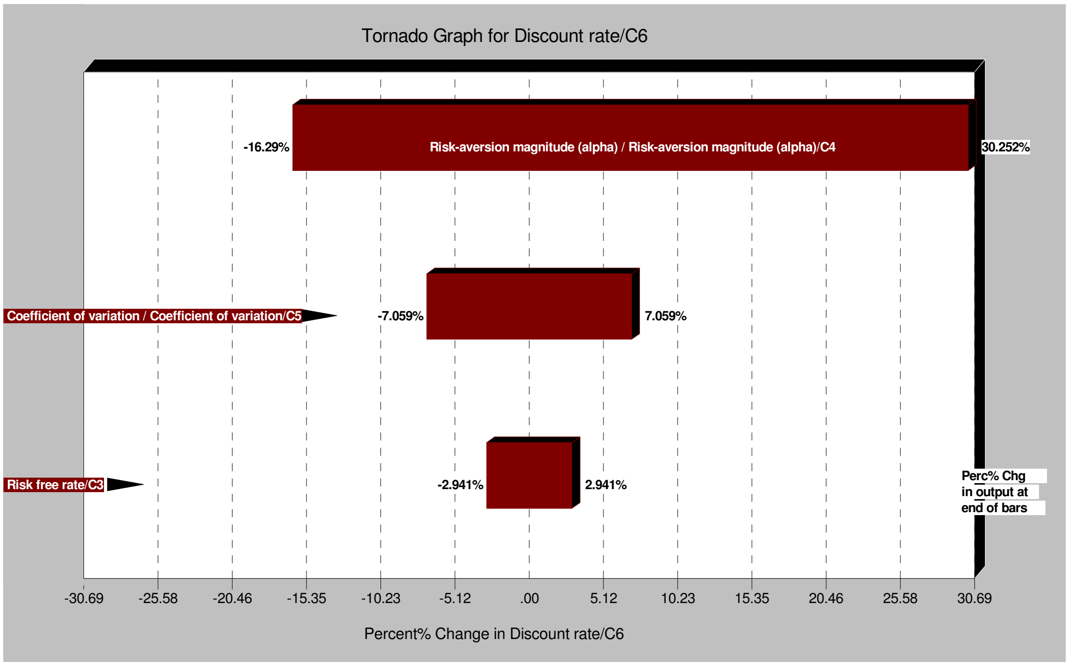
5.1 Analysis of the discount rate with quadratic preferences

Figure No 1 shows the sensitivity analysis conducted using the expression 16. As one may observe, the critical variable by far is the risk-aversion coefficient and it has an asymmetric impact over the estimated discount rate because a decrease of 10% in the alpha parameter increases the project's discount rate in more than 30%.

Figure No 2 shows simulated discount rates assuming a constant risk-free rate of 5%, a constant coefficient of variation equal to 0.15, and an alpha that is normally distributed with a mean of 2 and a standard deviation of 20%. As expected, the resulting discount rates tend to cluster around 20%. Although this is the lower bound, one may assume a different scenario and by getting different values for the parameters (especially for the coefficient of variation) one may obtain a higher subset of discount rates.

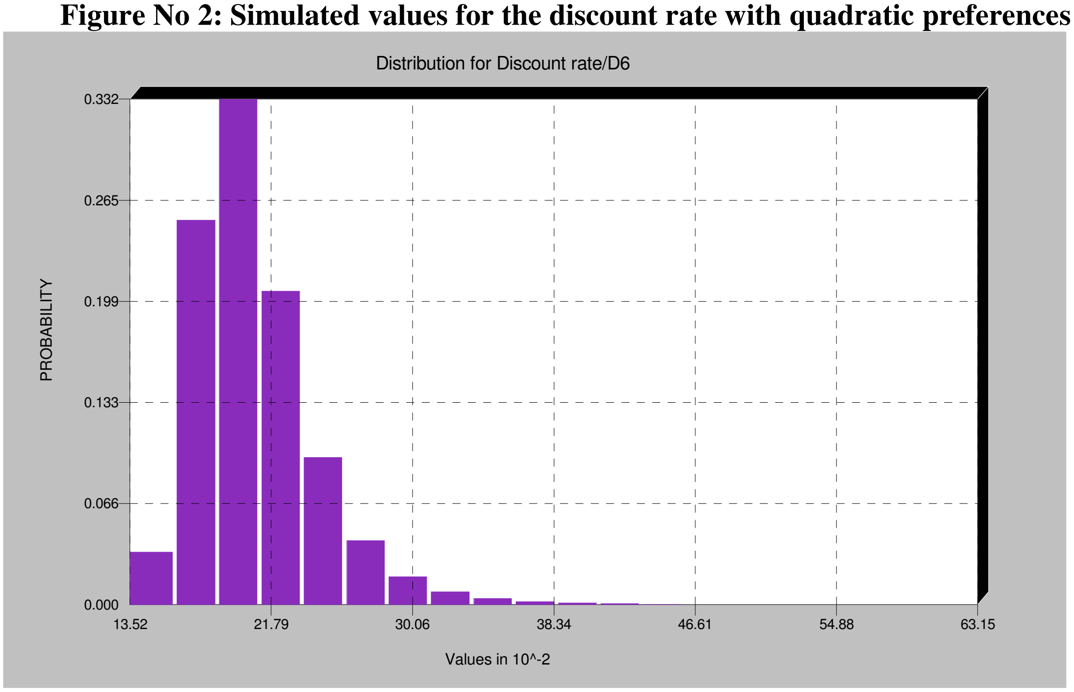
[9] Due to space limits, full proofs of equations 16 and 19 can be provided upon request

Figure No 1: Critical parameters for the estimation of discount rates with quadratic preferences



Source: Own elaboration

Figure No 2: Simulated values for the discount rate with quadratic preferences



Source: Own elaboration

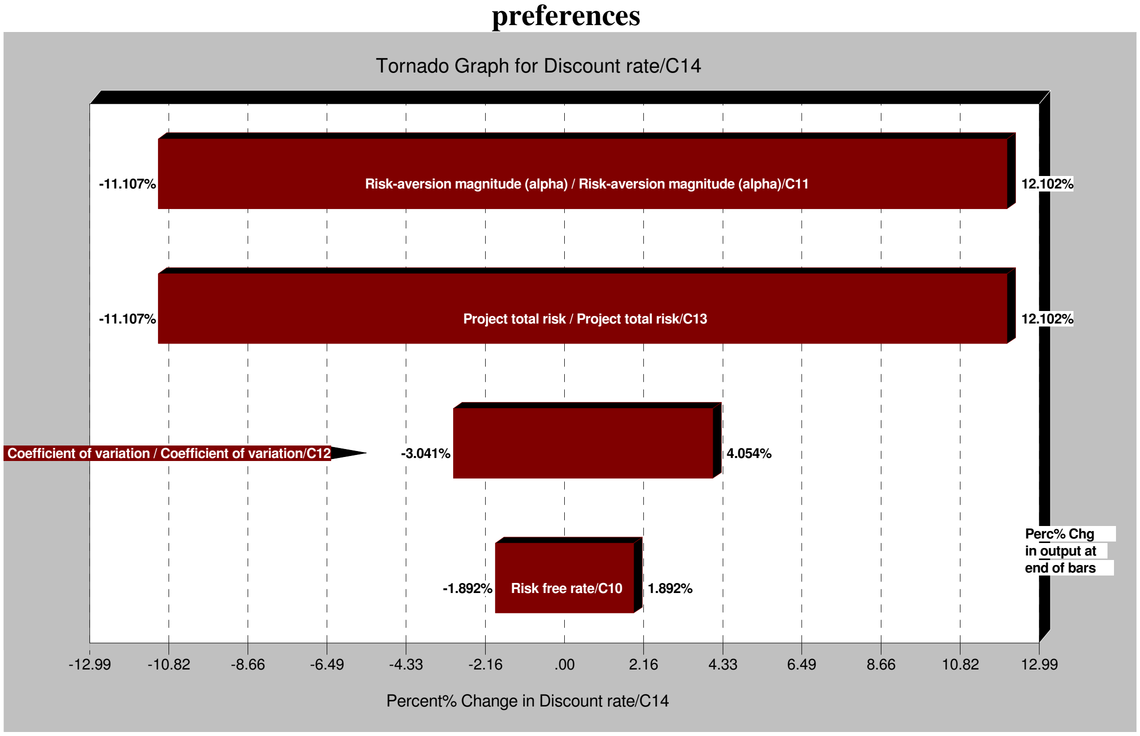
5.2 Analysis of the discount rate with logarithmic preferences

Figure No 3 shows the critical variables for this type of discount rate. The main conclusion is that the alpha and the project total risk are crucial parameters to estimate the project's discount rates. An increase of 10% in the risk aversion parameter alpha will decrease the project discount rate in 11%. It is interesting to note that although the alpha parameter is still important, its relative impact has been diminished.

Figure No 4 shows the simulation results for the discount rates with logarithmic preferences. Here, one uses a risk free rate of 5%, a coefficient of variation of 0.5, an alpha parameter that is normally distributed with a mean of 0.5 and a standard deviation of 10%, and the project total risk that is distributed according a uniform distribution with a lower value of 0.2 and an upper value of 0.4.

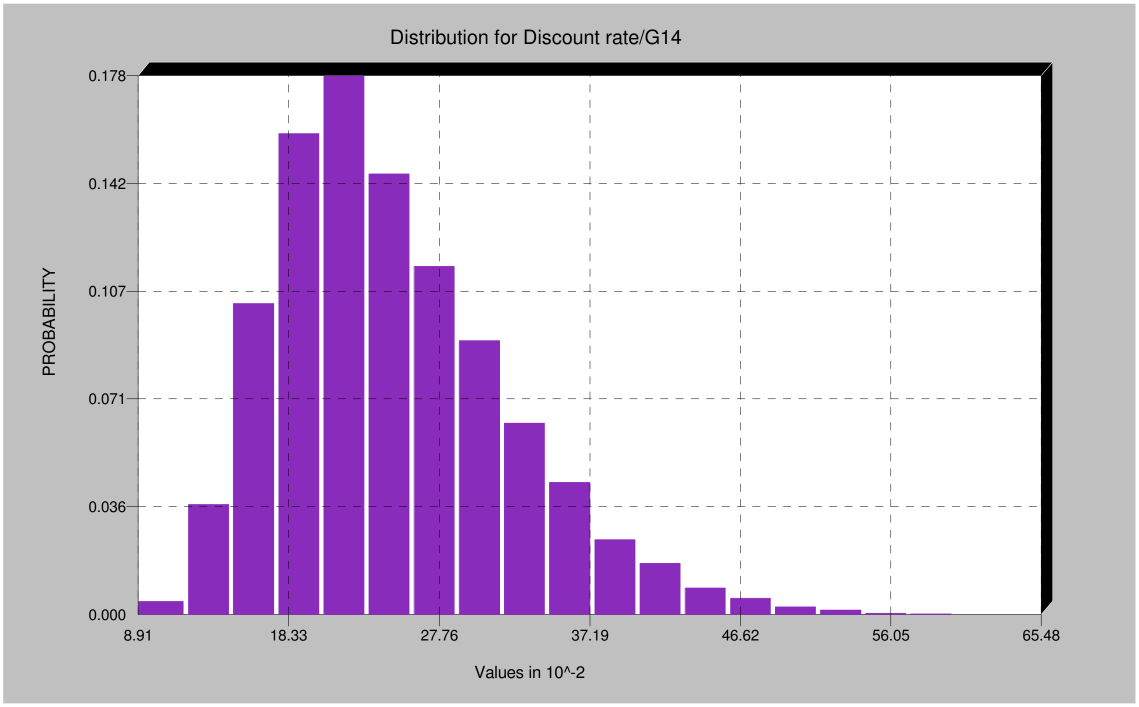
From this analysis one may conclude that the joint effect of both critical variables tend to increase the simulated values of the discount rates. However, the net effect is still asymmetric and it is skewed to the right as it should be by construction because they represent also the lower values.

Figure No 3: Critical parameters for the estimation of discount rates with logarithmic preferences



Source: Own elaboration

Figure No 4: Simulates values for the discount rate with logarithmic preferences



Source: Own elaboration

6. Conclusion

In this work one has derived subjective-based discount rates using the utility functions from the HARA family. In this way, one has obtained more generalized results than those of Mongrut and Ramirez (2006). Furthermore, these discount rate expressions have been derived assuming a less risk-averse entrepreneur who lacks a meaningful amount of initial wealth, and who puts all his money in undertaking his investment proposal or in running his firm, so he is non-diversified. One believes that this entrepreneur's profile is very common in emerging markets.

The utility specification applied in this work only includes pecuniary returns, such as the project or the firm's return on investment (ROI), and total risk. Depending on the specification, total risk is measured using the standard deviation and/or the coefficient of variation of the project's returns. In fact, one may include the non-pecuniary satisfactions related to the business, but in the prospective and risk analysis processes. These non-pecuniary satisfactions could include financial and operating project or firm's flexibility.

From the results, it seems that depending on the entrepreneur's risk preferences, the risk-aversion parameter alpha could be more or equally important as the measure of the project's total risk. However, given the assumptions, it seems more realistic to use the logarithmic specification instead of the quadratic one because the former assumes DARA. Given the importance of the entrepreneur's risk aversion, it is important that future work is directed toward improve the measurement of it. At the end of the day, what matters in seed capital project's valuation is not accuracy, but consistency.

7. References

Arrow K. (1965) Aspects of the Theory of Risk Bearing, Lectures Notes, Yrjö Jahnssonin Säätiö, Helsinki

Arrow K. (1971) Essays in the Theory of Risk Bearing, Markham, New York

Barthélémy F., T. Fisher, and J. Martel (2006) What Discount Rate Should Bankruptcy Judges Use? Estimates from Canadian Reorganization Data, THEMA Working Papers No 2006-19, Université de Cergy-Pontoise.

Breuer W. and M. Gürtler (2007) Two-Fund Separation and Hara Utility Reconsidered, April 3rd, available at SSRN: http://ssrn.com/abstract=659883

Breuer W. and M. Gürtler (2006) Performance Evaluation, Portfolio Selection, and HARA Utility, *European Journal of Finance*, Vol. 12, No 8, 649-669.

Cass D. and J. Stiglitz (1970) The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds, *Journal of Economic Theory,* Vol. 2, No 2, 122-160

Cameron A. and G. Gerdes (2003) Eliciting Individual-Specific Discount Rates, January 1st, University of Oregon Economics Working Paper No. 2003-10, available at SSRN: http://ssrn.com/abstract=436524

Connife D. (2006) The Flexible Three Parameter Utility Function. *Annals of Economics and Finance 1*, 57-63.

Erikson T. (2005) Idiosyncratic Portfolio Characteristics of Seed Venture Capital Funds. *Journal of Small Business and Enterprise Development*, Vol. 12, No 4, 545-555.

Fuenzalida D., S. Mongrut y M. Nash (2007) Evaluación de proyectos en mercados de capitales incompletos: el caso de los empresarios no diversificados. *El Trimestre Económico*, Vol. 74, No. 293, 69-121.

Gerber H. and G. Pafumi (1998) Utility Functions: from Risk Theory to Finance. *North American Actuarial Journal*, Vol. 2, No 3, 74-100.

Guu S. and J. Wang (2008) Zero-Level Pricing and the HARA Utility Functions, *Journal of Optimization Theory and Applications*, Springer Netherlands (Ed.) (Forthcoming).

James H. (1999) Owner as Manager, Extended Horizons and the Family Firm, *International Journal of the Economics of Business*, Vol. 6, No 1, 41-55.

Luenberger D. (2002) Arbitrage and Universal Pricing, *Journal of Economics Dynamics and Control*, Vol. 26, No.9, 1613–1628.

McInish T. and R. Kudla (1981) A New Approach to Capital Budgeting in Closely-Held Firms and Small Firms, *American Journal of Small Business*, Vol. 5, No 4, 30-36

McMahon R. and A. Stanger (1995) Understanding the Small Enterprise Financial Objective Function. *Entrepreneurship: Theory and Practice*, Vol. 19, No 4, 21-40.

Merton R. (1971) Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory*, Vol. 3, No 4, 373-413.

Meyer J. (2007) Representing Risk Preferences in Expected Utility Based Decision Models, *Annals of Operations Research,* Springer Netherlands (Ed.) (Forthcoming)

Menoncin F. (2002) Investment Strategies for HARA Utility Function: A General Algebraic Approximated Solution, Discussion Paper No. 2002-34, Université Catholique de Louvain.

Mongrut S. (2013) Valuation of Seed Capital Projects: A Prospective and Risk Analysis Perspective. Working Paper, Instituto Tecnológico y de Estudios Superiores de Monterrey (ITESM).

Mongrut S. and D. Ramirez (2006) Discount Rates for Emerging Capital Markets, *ICFAI Journal of Financial Economics*, Vol. 4. No 2, 35-55

Popova I. (2006) Scenario-Based and Mean-Variance Approaches to Portfolio Allocation with Hedge Funds, Working Paper, Albers School of Business and Economics, Seattle University.

Pratt J. (1964) Risk Aversion in the Small and in the Large, *Econometrica*, Vol. 32, No 1-2, 122-136.

Ross S. (1978) Mutual Fund Separation in Financial Theory – The Separating Distributions, *Journal of Economic Theory*, Vol. 17, No 2, 254-286.

Saha A. (1993) Expo-Power Utility: A Flexible Form for Absolute and Relative Risk Aversion. *American Journal of Agricultural Economics*, Vol. 75, No 4, 905-913.

Tobin J. (1958) Liquidity Preference as Behaviour towards Risk, *Review of Economic Studies*, Vol. 25, No 1, 65-86

Wei J., L. Cheng, and A. Lee (1999) Linear Conditional Expectation, Return Distributions, and Capital Asset Pricing Theories, *Journal of Financial Research*, Vol. 22, No 4, 471-487.

Xie D. (2000) Power Risk Aversion Utility Functions, *Annals of Economics and Finance 1*, 265-282.

Zurita F. (2005) Un examen a la tasa de descuento. *El Trimestre Económico*, Vol. 72, No 286, 257-281.